Implementation of the Goddard EOP Kalman Filter/Smoother

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Abstract

I describe the algorithms used in the Goddard EOP Kalman filter eop_kal.

1 Introduction

The orientation of the solid Earth changes because of external torques or the exchange of angular momentum between the solid Earth and the atmosphere and the oceans. The time evolution of this system can be viewed as obeying the following schematic linear equation:

$$\frac{d}{dt} EOP = F\, EOP + G\,(AAM + OAM) \tag{1}$$

here EOP are the EOP parameters, F is some transfer matrix, G is another transfer matrix, and AAM and OAM stand for, respectively, the atmosphere angular momentum and the ocean angular momentum. The goal of this note is to derive a model for EOP which accurately reproduces the spectrum of EOP. This model will then be used as a starting point for a Kalman filter.

In the following section we briefly review some facts from linear system theory. This is followed by the relevant equations for a continuous Kalman filter. In the following section we construct a model for UT1. This is followed by a model for PM. In much of the work which follows we need to evaluate functions of the form $\exp tA$ where A is a matrix. An appendix derives the explicit formula for the case where A is an arbitrary 2×2 matrix.

2 Linear Systems

2.1 Noiseless Case

Suppose that we have some model which is specified by a linear first order differential equation. This model can be written as:

$$\frac{d}{dt} X = FX \tag{2}$$

where X is a column vector, and F is a matrix. This equation is actually quite general, since any n-th order linear equation can be rewritten in this form, where the matrix is $n \times n$. Given initial conditions of the X at some time t_0 the solution to this equation is:

$$X(t) = \Phi(t - t_0)\; X(t_0) \tag{3}$$

where

$$\Phi(t) = \exp tF \tag{4}$$

$$\equiv I + \sum_{j=1}^{\infty} \frac{(tF)^j}{j!}. \tag{5}$$

We now turn to several examples.

2.1.1 Linear Motion

We consider the coupled differential equations

$$\begin{aligned} \frac{d}{dt}x &= \dot{x} \\ \frac{d}{dt}\dot{x} &= 0 \end{aligned} \tag{6}$$

which has the simple matrix form

$$\frac{d}{dt}\begin{pmatrix} x \\ \dot{x} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ \dot{x} \end{pmatrix} \tag{7}$$

The general solution to this is:

$$\begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix} = \Phi(t-t_0)\begin{pmatrix} x(t_0) \\ \dot{x}(t_0) \end{pmatrix}. \tag{8}$$

where

$$\Phi(t-t_0) = \exp t\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{9}$$

It is straightforward to verify that

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^j = 0 \quad j > 1. \tag{10}$$

Hence the only term that appears in the exponential sum is the $j = 1$ term:

$$\Phi(t) = I + t\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \tag{11}$$

It follows that:

$$\begin{aligned} x(t) &= x(t_0) + (t-t_0)\dot{x}(t_0) \\ \dot{x}(t) &= \dot{x}(t_0) \end{aligned} \tag{12}$$

2.1.2 Second order linear equation with constant coefficients

Consider the equation

$$\frac{d^2}{dt^2}x^2 + a\frac{d}{dt}x + bx = 0 \tag{13}$$

which is the general second order linear equation with constant coefficients. This can be re-written as:

$$\frac{d}{dt}\begin{pmatrix} x \\ \dot{x} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix}\begin{pmatrix} x \\ \dot{x} \end{pmatrix} \tag{14}$$

In an appendix we evaluate the exponential of an arbitrary 2×2 matrix. Suppose that $a_2 > \left(\frac{a_1}{2}\right)^2$. Then

$$\begin{aligned} \Phi(t) &= \exp t\begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix} & (15) \\ &= \exp(-\frac{a}{2}t) \times \left\{\cos\Omega t + \frac{\sin\Omega t}{\Omega}\begin{pmatrix} \frac{a}{2} & 1 \\ -b & \frac{-a}{2} \end{pmatrix}\right\} & (16) \end{aligned}$$

where

$$\Omega = \sqrt{b - \left(\frac{a}{2}\right)^2}. \tag{17}$$

The general solution for $x(t)$ is

$$\begin{aligned} x(t) &= \exp(-\frac{a}{2}t)\left\{\cos\Omega t\, x(t_0) + \frac{\sin\Omega t}{\Omega}\left(\frac{a}{2}x(t_0) + \dot{x}(t_0)\right)\right\} & (18) \\ \dot{x}(t) &= \exp(-\frac{a}{2}t)\left\{\cos\Omega t\, \dot{x}(t_0) - \frac{\sin\Omega t}{\Omega}(bx(t_0) + \frac{1}{2}a\dot{x}(t_0)\right\} \end{aligned}$$

2.2 Response of Linear Systems to Noise

In the presence of stochastic processes the initial equation is modified. A particularly simple modification is:

$$\frac{d}{dt}X = FX + G\xi(t) \tag{19}$$

where $\xi(t)$ is a vector which represents the noise source. The formal solution to this equation is given by:

$$X(t) = \Phi(t - t_0)X(t_0) + \int_{t_0}^{t} \Phi(t - \tau)G\xi(\tau)d\tau \tag{20}$$

Suppose that we are interested in the system after a long time, say an infinitely long time, and that

$$\Phi(t - (-\infty_0))X(-\infty) = 0 \tag{21}$$

This equation will be satisfied if initially the system is at rest, or if the transfer matrix is 0 for large time differences. In this case the system is driven entirely by the noise, and we have:

$$X(t) = \int_{-\infty}^{t} \Phi(t - \tau)G\xi(\tau)d\tau \tag{22}$$

If $\xi(t)$ can be expanded in a Fourier series

$$\xi(t) = \int_{-\infty}^{\infty} \exp i\omega t\, \xi(\omega) d\omega$$

then we can rewrite the expression for $X(t)$ as

$$\begin{aligned} X(t) &= \int_{-\infty}^{t} \Phi(t-\tau) G \int_{-\infty}^{\infty} \exp i\omega\tau\, \xi(\omega) d\omega d\tau \\ &= \int_{-\infty}^{\infty} \exp i\omega t \left\{ \int_{-\infty}^{t} \Phi(t-\tau) G \exp i\omega(\tau - t) \xi(\omega) d\tau \right\} d\omega \\ &= \int_{-\infty}^{\infty} \exp i\omega t \Phi(\omega) G \xi(\omega) d\omega \end{aligned} \tag{23}$$

where

$$\Phi(\omega) = \int_{0}^{\infty} \exp i\omega t \Phi(t) d\omega \tag{24}$$

It follows from this that the spectral density of $X(t)$ is:

$$P_X(\omega) = |\Phi(\omega) G \xi(\omega)|^2 \tag{25}$$

Of course this depends on equation (24) existing. If this is not the case, then direct means may work.

2.3 Examples

First order system driven by noise For this system we have

$$\begin{aligned} x(t) &= \int_{t_0}^{t} \xi(\tau) d\tau = \int_{t_0}^{t} \left\{ \int_{-\infty}^{\infty} \xi(\omega) \exp i\omega\tau d\omega \right\} d\tau \\ &= \int_{-\infty}^{\infty} \left\{ \int_{t_0}^{t} \xi(\omega) \exp i\omega\tau d\tau \right\} d\omega \\ &= \int_{-\infty}^{\infty} \xi(\omega) \frac{1}{\omega} \left(\exp i\omega t - \exp i\omega t_0 \right) d\omega \end{aligned} \tag{26}$$

The spectral density is then:

$$P_X(\omega) = |\xi(\omega)|^2 \frac{1}{\omega^2} \tag{27}$$

2.3.1 Second order system driven by white noise.

Suppose we take the second order system previously studied, and modify it so that we have

$$\frac{d}{dt}\begin{pmatrix} x \\ \dot{x} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix}\begin{pmatrix} x \\ \dot{x} \end{pmatrix} + \begin{pmatrix} 0 \\ \xi(t) \end{pmatrix} \tag{28}$$

To find the spectrum of $x(t)$ we need:

$$\begin{aligned} \Phi(\omega) &= \int_0^\infty \exp i\omega t \exp(-\gamma t)\frac{\sin \Omega t}{\Omega} dt && (29) \\ &= \frac{1}{\gamma - i(\omega+\Omega)}\frac{1}{\gamma - i(\omega-\Omega)} && (30) \end{aligned}$$

where I have substituted $\gamma = \frac{a_1}{2}$. Hence

$$\begin{aligned} P_X(\omega) &= |\xi(\omega)|^2 |\Phi(\omega)|^2 && (31) \\ &= |\xi(\omega)|^2 \frac{1}{(\gamma^2+\omega^2+\Omega^2)^2 - 4\omega^2\Omega^2} && (32) \end{aligned}$$

The term $|\Phi(\omega)|^2$ has a peak at

$$\omega_0 = \sqrt{\Omega^2 - \gamma^2} \tag{33}$$

with magnitude $1/(4\Omega^2\gamma^2)$. Expanding this term in a Taylor series about the peak we find

$$|\Phi(\omega)|^2 = \frac{1}{4\Omega^2\gamma^2}\left\{1 - (\Delta\omega)^2 \frac{\Omega^2-\gamma^2}{\Omega^2\gamma^2}\right\} + O\left(\Delta\omega^3\right) \tag{34}$$

which implies that the half-width is

$$\begin{aligned} \Delta\omega_{\frac{1}{2}} &= \sqrt{\frac{2\gamma^2}{1-\frac{\gamma^2}{\Omega^2}}} \\ &\simeq \sqrt{2}\gamma && (35) \end{aligned}$$

These relations allow us to deduce the coefficients a and b.

3 Kalman Filtering

Typically what happens is that we have some measurements (with uncertainties) of some of the components of the $X(t_j)$ at different times t_j. We want to use these measurements to obtain optimal estimates of the $X(t_k)$.

Suppose that we have an initial estimate $\hat{X}(t)$ with covariance $P_{\hat{X}}$ and a measurement $Z(t)$ with covariance P_Z. The new estimate $\hat{X}^+(t)$ is given by:

$$\hat{X}^+(t) = \frac{1}{P_{\hat{X}}^{-1} + P_Z^{-1}}\left(P_{\hat{X}}^{-1}\hat{X} + P_Z^{-1} Z\right) \tag{36}$$

with covariance

$$P^+_{\hat{X}}(t) = \frac{1}{P^{-1}_{\hat{X}} + P^{-1}_{Z}}. \tag{37}$$

The optimal estimates $\hat{X}^+(t)$ at a later time t_f are easily found from above:

$$\hat{X}^+(t_f) = \Phi(t_f - t)\hat{X}^+(t) \tag{38}$$

and the covariance at some later time t_f is given by

$$P^+_{\hat{X}}(t_f) = \Phi(t_f - t)P^+_{\hat{X}}(t)\Phi^T(t_f - t) \tag{39}$$

These equations are essentially what are known as the "Kalman Filter" equations.

3.1 Linear Equation With Noise Input

If we are trying to estimate the $X(t)$, we use the same equations as before. The only equation which is modified is the equation for the covariance, which becomes:

$$P^+_{\hat{X}}(t_f) = \Phi(t_f - t)P^+_{\hat{X}}(t)\Phi^T(t_f - t) + \int_t^{t_f} \Phi(t_f - \tau)G\xi(\tau)\xi^T(\tau)G^T\Phi^T(t_f - \tau) \tag{40}$$

If the stochastic processes are uncorrelated, then we can replace $\xi(\tau)\xi^T(\tau)$ by a diagonal matrix.

4 Model for UT1

In the absence of external perturbations, UT1 should evolve linearly with time. Therefore, our starting point for UT1 is:

$$\frac{d}{dt}\begin{pmatrix} UT1 \\ -LOD \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} UT1 \\ -LOD \end{pmatrix} \tag{41}$$

which has the matrix solution:

$$\begin{pmatrix} UT1(t) \\ -LOD(t) \end{pmatrix} = \begin{pmatrix} 1 & (t - t_0) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} UT1(t_0) \\ -LOD(t_0) \end{pmatrix} \tag{42}$$

This can be recast in the more familiar form:

$$UT1(t) = UT1(t_0) - (t - t_0)LOD(t_0) \tag{43}$$

$$LOD(t) = LOD(t_0) \tag{44}$$

i .e., $UT1$ evolves linearly with time. These equations are written with $-LOD$ because what *solve* reports is $\frac{dUT1}{dt} = -LOD$. When we come to implementing the filter it is easier to work with $\frac{dUT1}{dt} = -LOD$: we don't have to switch the sign of$\frac{dUT1}{dt}$, or change the sign of various terms in the correlation matrix.

Figure 1 is a plot of the power spectrum of UT1 derived from VLBI data. At high frequencies, say under 30 days, the spectrum is well approximated by:

$$\begin{aligned} P_{UT1}(T) &= \left(\frac{T}{Day}\right)^2 10^{-4} ms^2/CPD \\ &= \left(\frac{2\pi}{\omega}\right)^2 10^{-4} ms^2/CPD = \frac{1}{\omega^2} 0.0039 ms^2/CPD \end{aligned} \tag{45}$$

Based on our analysis above, this is recognized as the power spectrum you would obtain by integrating white noise with spectral density of $0.0039ms^2/D^3$. This suggests that the simple model for UT1 be modified to:

$$\begin{pmatrix} UT1 \\ -LOD \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} UT1 \\ -LOD \end{pmatrix} + \begin{pmatrix} 0 \\ \xi_L \end{pmatrix} \tag{46}$$

where ξ_L has a uniform spectral density of $0.0039ms^2/D^3$.

A closer look at Figure 1 shows that there are prominent peaks at the annual and semi-annual frequencies. If we want to include these peaks we need to the model for LOD more complicated. In particular, we write:

$$\xi_L = \omega_L + A_L + S_L \tag{47}$$

where ω_L is white noise, and A_L and S_L are annual and semi-annual seasonal terms which have a harmonic time dependence. We already know how to construct a model driven by noise which has a harmonic time dependence. This was done in ***. All we need to do is pick the coefficients based on the location of the peaks and their width.

Out total model for UT1 is then given as:

$$\frac{d}{dt} X = FX + \omega \tag{48}$$

where

$$X = \begin{pmatrix} UT1 \\ -LOD \\ A_L \\ \dot{A}_L \\ S_L \\ \dot{S}_L \end{pmatrix} \quad F = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -b_{UA} & -a_{UA} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -b_{US} & -a_{US} \end{pmatrix} \quad \omega = \begin{pmatrix} 0 \\ \omega_L \\ 0 \\ \omega_{UA} \\ 0 \\ \omega_{US} \end{pmatrix} \tag{49}$$

5 Building a Model for Polar Motion

In the absence of noise, a simple model which describes polar motion is given by:

$$\frac{d}{dt} \begin{pmatrix} X \\ Y \end{pmatrix} = \sigma \begin{pmatrix} -\frac{1}{2Q} & \sigma \\ -1 & -\frac{1}{2Q} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \tag{50}$$

The matrix on the RHS of this can be exponentiated to give:

$$\Phi(t) = \exp(-t\frac{\sigma}{2Q}) \begin{pmatrix} \cos\sigma t & \sin\sigma t \\ -\sin\sigma t & \cos\sigma t \end{pmatrix} \tag{51}$$

where σ is the frequency of the chandler wobble, and Q is the quality factor. In the presence of AAM and OAM the simple PM model is modified to:

$$\frac{d}{dt}\begin{pmatrix} X \\ Y \end{pmatrix} = \sigma \begin{pmatrix} -\frac{1}{2Q} & 1 \\ -1 & -\frac{1}{2Q} \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} + \sigma \begin{pmatrix} \frac{1}{2Q} & 1 \\ 1 & -\frac{1}{2Q} \end{pmatrix}\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \tag{52}$$

In analogy with the case for UT1, we also add a seasonal term to PM. It is well known that the pole also experiences a long term drift. This suggests that we add linear terms as well. In this case the total model for PM is given by:

$$X = \begin{pmatrix} PM \\ PX \\ \mu_1 \\ \mu_2 \\ S \\ \dot{S} \\ dPM \\ dPY \end{pmatrix} \quad F = \begin{pmatrix} -\gamma & \sigma & \gamma & \sigma & \sigma & 0 & 1 & 0 \\ -\sigma & -\gamma & \sigma & -\gamma & -\gamma & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_2 & -a_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \omega = \begin{pmatrix} 0 \\ 0 \\ \omega_l \\ \omega_l \\ 0 \\ \omega_A \\ \omega_L \\ \omega_L \end{pmatrix} \tag{53}$$

5.1 Evaluation of $\Phi(t) = \exp tF$

We now turn to the evaluation of $\Phi(t)$. The F matrix for both UT1 and polar motion takes the general form:

$$F = \begin{pmatrix} A & B_1 & B_2 & ... \\ 0 & C_1 & 0 & \\ 0 & 0 & C_2 & \\ 0 & 0 & ... & \end{pmatrix} \tag{54}$$

where A, B_j, C_j, are sub-matrices. We want to evaluate:

$$\exp tF = \sum_{n=0}^{\infty} \frac{tF^n}{n!} \tag{55}$$

Theorem 1

$$F^n = \begin{pmatrix} A^n & \sum_{j=0}^{n-1} A^{n-1-j} B_1 C_1^j & \sum_{j=0}^{n-1} A^{n-1-j} B_2 C_2^j & ... \\ 0 & C_1^n & 0 & 0 \\ 0 & 0 & C_2^n & 0 \\ 0 & 0 & 0 & \end{pmatrix} \quad n > 0 \tag{56}$$

This is true for $n = 1$. I now show that if it is true for n it will also be true for $n + 1$.

$$FF^n = \begin{pmatrix} A^{n+1} & A\sum_{j=0}^{n-1} A^{n-1-j} B_1 C_1^j + B_1 C_1^n & A\sum_{j=0}^{n-1} A^{n-1-j} B_2 C_2^j + B_1 C_1^n & ... \\ 0 & C_1^{n+1} & 0 & 0 \\ 0 & 0 & C_2^{n+1} & 0 \\ 0 & 0 & 0 & \end{pmatrix} \tag{57}$$

$$= \begin{pmatrix} A^{n+1} & \sum_{j=0}^{n} A^{n-j} B_1 C_1^j & \sum_{j=0}^{n} A^{n-j} B_2 C_2^j & \dots \\ 0 & C_1^{n+1} & 0 & 0 \\ 0 & 0 & C_2^{n+1} & 0 \\ 0 & 0 & 0 & \end{pmatrix} \tag{58}$$

which completes the proof.

From this it follows that

$$\exp tF = \begin{pmatrix} \exp tA & G_{AB_1C_1}(t) & G_{AB_2C_2}(t) & \dots \\ 0 & \exp tC_1 & 0 & 0 \\ 0 & 0 & \exp tC_2 & 0 \\ 0 & 0 & 0 & \end{pmatrix} \tag{59}$$

where

$$G_{AB_1C_1}(t) = \sum_{n=1}^{\infty} \frac{t^n}{n!} \sum_{j=0}^{n-1} A^{n-j-1} B_1 C_1^j \tag{60}$$

The hard part consists of evaluating this matrix function. It may happen that this can be evaluated by special tricks which depend on the form of the matrices which make it up. This turns out to be the case in most of what we do. However, there are also two general ways to evaluate it, to which we turn now.

5.2 G_{ABC} as an integral.

Consider the differential equation which defines $\exp tF$:

$$\frac{d}{dt} \exp tF = F \exp tF \tag{61}$$

If we look at the off-diagonal terms we get the following equations:

$$\frac{d}{dt} G_{ABC}(t) = A G_{ABC}(t) + B \exp tC \tag{62}$$

where for simplicity I have neglected the subscripts on B_1 and C_1 which has the formal solution:

$$G_{ABC}(t_0) = e^{tA}\ G_{ABC}(t_0) + \int_{t_0}^{t} e^{(t-\tau)A}\ B\ e^{-\tau C}\ d\tau \tag{63}$$

Since $G_{ABC}(0) = 0$, this can be simplified to yield:

$$G_{ABC}(t) = \int_{t_0}^{t} e^{(t-\tau)A}\ B\ e^{-\tau C} d\tau \tag{64}$$

Note that I have used no special conditions on the matrices A, B, or C. Hence, if we can evaluate $\exp tA$ and $\exp tC$ then we can evaluate $G_{ABC}(t)$.

5.3 Derivation of G_{ABC} in terms of eigenvectors of A.

Assume that the matrix A has a complete set of eigenvectors and eigenvalues:

$$Av_a = \lambda_a v_a \tag{65}$$

Then we can decompose each of the columns of B into these eigenvectors. Gathering all of the eigenvectors of a common eigenvalue together, we decompose B into "eigen-matrices".

$$\begin{aligned} B &= \sum B_a \\ AB_a &= \lambda_a B_a \end{aligned} \tag{66}$$

Consider a single term in the sum that defines G_{ABC}:

$$\sum_{j=0}^{n-1} A^{n-1-j} B C^j = \sum_a \sum_{j=0}^{n-1} \lambda_a^{n-1-j} B_a C^j \tag{67}$$

Now

$$\begin{aligned} \sum_{j=0}^{n-1} \lambda_a^{n-1-j} B_a C^j &= \lambda_a^{n-1} B_a \sum_{j=0}^{n-1} \left(\frac{C}{\lambda_a}\right)^j \\ &= \lambda_a^{n-1} B_a \frac{1-\left(\frac{C}{\lambda_a}\right)^n}{1-\frac{C}{\lambda_a}} \\ &= B_a \frac{\lambda_a^n - C^n}{\lambda_a - C} \end{aligned} \tag{68}$$

From this it follows that

$$G_{ABC}(t) = \sum_a B_a \frac{1}{\lambda_a - C} \left(\exp t\lambda_a - \exp tC\right) \tag{69}$$

5.4 Calculation of Noise Covariance

We now turn to the calculation of the noise covariance. The formula for this is:

$$Cov(t_f - t_i) = \int_{t}^{t_f} \Phi(t_f - \tau) Q \Phi^T(t_f - \tau) \tag{70}$$

where Q is a diagonal matrix. If our transition matrix has the general form given above, then the integrand can be written as:

$$\begin{pmatrix} e^{tA} Q_0 e^{tA^T} + \sum_j G_{AB_jC_j}(t) Q_j G^T_{AB_jC_j} & G^T_{AB_1C_1}(t) Q_1 e^{tC_1^T} & G^T_{AB_2C_2}(t) Q_2 e^{tC_2^T} & ... \\ e^{tC_1}\, Q_1\, G^T_{AB_1C_1}(t) & e^{tC_1} Q_1 e^{tC_1^T} & 0 & 0 \\ e^{tC_2}\, Q_2\, G^T_{AB_2C_2}(t) & 0 & e^{tC_2} Q_2 e^{tC_2^T} & 0 \\ ... & 0 & & \end{pmatrix} \tag{71}$$

where Q_j is the diagonal part of Q restricted to appropriate subspace.

6 Evaluation of $\Phi(t)$ for UT1.

We now explicitly evaluate the transition matrix for UT1. For this case we have:

$$\begin{aligned} A &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ B_1 &= B_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned} \tag{72}$$

and D and E are the seasonal matrices. We also need the following identities:

$$\begin{aligned} A^j &= 0 \quad j \geq 2 \\ AB_k &= 0 \end{aligned} \tag{73}$$

From which it follows that:

$$\sum_{j=0}^{n-1} A^{n-j-1} BC^j = BC^{n-1} \quad n > 1 \tag{74}$$

Therefore the off-diagonal term involving B which appears in $\exp tF$ takes the form:

$$\begin{aligned} G_{ABC}(t) &= \sum_{n=1}^{\infty} Bt^n \frac{C^{n-1}}{n!} \\ &= \sum_{n=1}^{\infty} Bt^n \frac{C^n}{n!} D^{-1} \\ &= B\left(\exp tC - 1\right) C^{-1} \end{aligned} \tag{75}$$

Note that

$$C^{-1} = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix}^{-1} = \frac{1}{b} \begin{pmatrix} -a & -1 \\ b & 0 \end{pmatrix} \tag{76}$$

hence

$$BC^{-1} = \frac{1}{b} \begin{pmatrix} -a & -1 \\ 0 & 0 \end{pmatrix} \tag{77}$$

Putting everything together we find:

$$\Phi(t) = \begin{pmatrix} e^{tA} & BC_{UA}^{-1}\left(e^{tC_{UA}} - 1\right) & BC_{UA}^{-1}\left(e^{tC_{UA}} - 1\right) \\ 0 & e^{tC_{UA}} & 0 \\ 0 & 0 & e^{tC_{US}} \end{pmatrix} \tag{78}$$

where each block is a 2x2 sub-matrix, and

$$e^{tA} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \tag{79}$$

$$e^{tC_{UA}} = \exp(-\frac{a_{UA}}{2}t) \times \left\{ \cos\Omega t + \frac{\sin\Omega t}{\Omega} \begin{pmatrix} \frac{a_{UA}}{2} & 1 \\ -b_{UA} & \frac{-a_{UA}}{2} \end{pmatrix} \right\} \tag{80}$$

6.0.1 Calculation of Noise Covariance for UT1

The general form for the noise-covariance matrix was given above. In this section we evaluate it for UT1. For UT1 the diagonal matrices all have the same form:

$$Q_j = diag(0, q_j) \tag{81}$$

hence:

$$\exp tA\, Q_w\, \exp tA^T = q_w \begin{pmatrix} (\exp tA)_{12}^2 & (\exp tA)_{12}(\exp tA)_{22} \\ (\exp tA)_{12}(\exp tA)_{22} & (\exp tA)_{22}^2 \end{pmatrix} \tag{82}$$

with a similar formula for $\exp tC_{UA}\, Q_{UA}\, \exp tC_{UA}^T$ etc. Continuing with the diagonal terms, we have:

$$\exp tA\, Q_w\, \exp tA^T = q_w \begin{pmatrix} t^2 & t \\ t & 1 \end{pmatrix}$$

which can easily be integrated:

$$\int \exp tA\, Q_w\, \exp tA^T dt = q_w \begin{pmatrix} \frac{t^3}{3} & \frac{t^2}{2} \\ \frac{t^2}{2} & t \end{pmatrix} \tag{83}$$

The seasonal terms are somewhat more complicated:

$$e^{tC}\, Qe^{tC^T} = q\exp(-at) \begin{pmatrix} \left(\frac{\sin wt}{w}\right)^2 & \cdots \\ \left(\frac{\sin wt}{w}\right)\left(\cos wt - \frac{a}{2}\frac{\sin wt}{w}\right) & \left(\cos wt - \frac{a}{2}\frac{\sin wt}{w}\right)^2 \end{pmatrix} \tag{84}$$

which integrates to:

$$\begin{aligned} \int_0^t \exp(-a\tau)\left(\frac{\sin w\tau}{w}\right)^2 d\tau &= -\frac{1}{2}\frac{1}{aw^2}e^{-at} + \frac{2}{a\,(a^2+4w^2)} + e^{-at}\frac{a\cos 2wt - 2w\sin 2wt}{2w^2\,(a^2+4w^2)} \\ &= -\frac{1}{2}\frac{1}{aw^2}e^{-at} + \frac{2}{4ab} + e^{-at}\frac{a\cos 2wt - 2w\sin 2wt}{8w^2 b} \end{aligned} \tag{85}$$

$$\int_0^t \exp(-a\tau)\left(\frac{\sin w\tau}{w}\right)(\cos w\tau - \frac{a}{2}\frac{\sin w\tau}{w})d\tau = -\frac{1}{4}\left(\cos 2wt - 1\right)\frac{e^{-at}}{w^2} \tag{86}$$

$$\begin{aligned} \int_0^t \exp(-a\tau)(\cos w\tau - \frac{a}{2}\frac{\sin w\tau}{w})^2 d\tau &= \frac{1}{2a} - \frac{1}{8w^2a}e^{-at}(4w^2+a^2) + e^{-at}\frac{a\cos 2wt + 2w\sin 2wt}{8w^2} \\ &= \frac{1}{2a} - \frac{b}{2w^2a}e^{-at} + e^{-at}\frac{a\cos 2wt + 2w\sin 2wt}{8w^2} \end{aligned} \tag{87}$$

where I have made repeated use of the identity

$$(4w^2 + a^2) = 4b. \tag{88}$$

6.0.2 Evaluation of $G_{ABC}(t)QG^T_{ABC}$ for UT1

The noise covariance matrix for UT1 has two terms of this form:

$$B\left(\exp tC-1\right)C^{-1}Q\left[B\left(\exp tC-1\right)C^{-1}\right]^T \tag{89}$$

Note that

$$\begin{aligned} C^{-1}QC^{-1T} &= \frac{1}{b}\begin{pmatrix} -a & -1 \\ b & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix}\frac{1}{b}\begin{pmatrix} -a & b \\ -1 & 0 \end{pmatrix} \\ &= \frac{q}{b^2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned} \tag{90}$$

Combining this with the explicit form of B above, we get:

$$G_{ABC}(t)QG^T_{ABC}(t) = \frac{q}{b^2}\left[\exp tC-1\right]^2_{11}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{91}$$

The integral we need to evaluate is the term in square brackets:

$$\int_0^t \left((\cos w\tau + \frac{a}{2}\frac{\sin w\tau}{w})\exp(-\tau a/2) - 1\right)^2 d\tau \tag{92}$$

which I evaluated using maple. The result, after simplification, is:

$$\begin{aligned} &\frac{1}{2}\frac{(4w^2-11a^2)}{(a^2+4w^2)\,a}+t \\ &+e^{-ta}\frac{1}{8}\frac{\left(-(4w^2+a^2)^2+a^2(\cos 2wt)\left(-12w^2+a^2\right)+wa\left(\sin 2wt\right)\left(8w^2-6a^2\right)\right)}{(a^2+4w^2)\,aw^2} \\ &+e^{-\frac{1}{2}ta}\frac{\left(2aw\left(\sin wt\right)\left(a^2-4w^2\right)+8a^2w^2\cos wt\right)}{(a^2+4w^2)\,aw^2} \end{aligned} \tag{93}$$

$$\begin{aligned} = &\frac{1}{2}\frac{(4w^2-11a^2)}{(a^2+4w^2)\,a}+t \\ &+e^{-ta}\frac{1}{8}\frac{1}{aw^2}\left(-4b+\frac{a^2(\cos 2wt)\left(-12w^2+a^2\right)+wa\left(\sin 2wt\right)\left(8w^2-6a^2\right)}{4b}\right) \\ &+e^{-\frac{1}{2}ta}\frac{1}{4baw^2}\left(2aw\left(\sin wt\right)\left(a^2-4w^2\right)+8a^2w^2\cos wt\right) \end{aligned} \tag{94}$$

6.0.3 Evaluation of $G_{ABC}(t)Qe^{tC^T}$ for UT1

The noise covariance matrix for UT1 has two terms of this form:

$$B\left(e^{tC}-1\right)C^{-1}Qe^{tC^T} \tag{95}$$

Using the explicit form of the matrices, we find

$$B\left(e^{tC}-1\right)C^{-1}Q = \frac{-q}{b}\left[e^{tC}-1\right]_{11}\begin{pmatrix} \left[e^{tC}\right]_{12} & \left[e^{tC}\right]_{22} \\ 0 & 0 \end{pmatrix} \tag{96}$$

The integral of the first term is:

$$\begin{aligned}
&\int_0^t \left((\cos w\tau + \frac{a}{2}\frac{\sin w\tau}{w}) \exp(-\tau a/2) - 1 \right) \exp(-\tau a/2) \left(\frac{\sin wt}{w} \right) d\tau \\
= \;& \frac{1}{4}\frac{1}{(a^2+w^2)\, w^2 a} e^{-at} \left(a\left(a^2 - 2w^2\right)\cos 2wt - 3wa^2 \sin 2wt + a(2w^2 - a^2) \right) \\
& + \frac{1}{4}\frac{1}{(a^2+w^2)\, w^2 a} \left(8e^{-\frac{1}{2}at} (\sin wt)\, wa^2 - 2wa^2 \sin wt - 8 (\sin wt)\, w^3 \right)
\end{aligned} \tag{97}$$

The second term is:

$$\begin{aligned}
&\int_0^t \left((\cos w\tau + \frac{a}{2}\frac{\sin w\tau}{w}) \exp(-\tau a/2) - 1 \right) (\cos w\tau - \frac{a}{2}\frac{\sin w\tau}{w}) \exp(-\tau a/2) d\tau \\
= \;& -\frac{1}{8}\frac{e^{-at}(4w^2 - a^2 - 2wa \sin 2wt + a^2 \cos 2wt) + 8e^{-\frac{1}{2}at} wa \sin wt - 4w^2}{aw^2}
\end{aligned} \tag{98}$$

6.1 Evaluation of $\Phi(t) = \exp tF$ for Polar Motion

We now turn to the evaluation of the transition matrix for polar motion. From above, this takes the general form:

$$\exp tF = \begin{pmatrix} e^{tA} & G_{AB_1C_1}(t) & G_{AB_2C_2}(t) & G_{AB_3C_3}(t) \\ 0 & e^{tC_1} & 0 & 0 \\ 0 & 0 & e^{tC_2} & 0 \\ 0 & 0 & 0 & e^{tC_3} \end{pmatrix} \tag{99}$$

Evaluation of $G_{AB_1C_1}(t)$ Since the matrix C_1 is identically zero, it is straightforward to verify that

$$\begin{aligned}
G_{AB_1C_1}(t) &= \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{j=0}^{n-1} A^{n-j} B_1 C_1^j \\
&= \sum_{n=1}^{\infty} \frac{A^{n-1} t^n}{n!} B_1 \\
&= (\exp tA - 1) A^{-1} B_1
\end{aligned} \tag{100}$$

Using the explicit form of A and B given above, we find:

$$A^{-1}B = \frac{1}{\gamma^2 + \sigma^2} \begin{pmatrix} -\gamma & -\sigma \\ \sigma & -\gamma \end{pmatrix} \begin{pmatrix} \gamma & \sigma \\ \sigma & -\gamma \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{101}$$

hence

$$G_{AB_1C_1}(t) = (\exp tA - 1) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{102}$$

Evaluation of $G_{AB_2C_2}(t)$ The other off diagonal term is much more difficult to evaluate. We use the method of decomposition in terms of eigenvalues. The matrix A has two eigenvalues: $\lambda_\pm = (-\gamma \pm i\sigma)$ with eigenvectors $(1, \pm i)$:

$$\begin{pmatrix} -\gamma & \sigma \\ -\sigma & -\gamma \end{pmatrix} \begin{pmatrix} 1 \\ \pm i \end{pmatrix} = (-\gamma \pm i\sigma) \begin{pmatrix} 1 \\ \pm i \end{pmatrix} \tag{103}$$

We decompose the matrix B_2 into eigen-matrices corresponding to the eigenvalues.

$$\begin{aligned} B_2 &= B_+ + B_- \\ &= \frac{\sigma - i\gamma}{2} \begin{pmatrix} 1 & 0 \\ i & 0 \end{pmatrix} + \frac{\sigma + i\gamma}{2} \begin{pmatrix} 1 & 0 \\ -i & 0 \end{pmatrix} \end{aligned} \tag{104}$$

From this it follows that

$$G_{AB_2C_2}(t) = B_+ \frac{1}{\lambda_+ - C} \left(\exp t\lambda_+^n - \exp tC_2\right) + B_- \frac{1}{\lambda_- - D} \left(\exp t\lambda_-^n - \exp tC_2\right) \tag{105}$$

For our case we have $\lambda_- = \lambda_+^*$ and $B_- = B_+^*$. Hence this equation can be re-written as:

$$G_{AB_2C_2}(t) = 2 \times \mathrm{Re}\, B_+ \frac{1}{\lambda_+ - C_2} \left(\exp t\lambda_+^n - \exp tC_2\right) \tag{106}$$

Evaluation of $G_{AB_3C_3}(t)$ Since C_3 is identically zero, the same considerations as above apply, and we have

$$\begin{aligned} G_{AB_3C_3}(t) &= (\exp tA - 1)A^{-1}B_3 \\ &= (\exp tA - 1)A^{-1} \end{aligned} \tag{107}$$

where I have used the fact that B_3 is zero.

6.2 Evaluation of Noise Covariance for Polar Motion

The general form of the covariance matrix was given above. The diagonal matrix Q takes the form:

$$Q = diag(0, 0, q_\mu, q_\mu, 0, q_s, q_L, q_L) \tag{108}$$

Since the first two elements are 0, the top corner of the noise covariance matrix is given by:

$$\sum_{j=1}^{3} G_{AB_jC_j}(t) Q_j G^T_{AB_jC_j} \tag{109}$$

The first term in this is:

$$G_{AB_1C_1}(t) Q_1 G^T_{AB_1C_1} = (e^{tA} - 1) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} (e^{tA^T} - 1) \tag{110}$$

$$= \left(e^{-2t\gamma} + 1 - 2e^{-\gamma\tau} \cos \sigma t\right) I \tag{111}$$

This can be easily integrated to yield:

$$\int_0^t \left(e^{-2\tau\gamma}+1-2e^{-\gamma\tau}\cos(\sigma\tau)\right)\,d\tau \qquad (112)$$
$$= t+\frac{1-e^{-2t\gamma}}{2\gamma} \quad +2\frac{e^{-\gamma t}\gamma\cos\sigma t-e^{-\gamma t}\sigma\sin\sigma t-\gamma}{\gamma^2+\sigma^2}$$

Similarly, the last term in the top takes the form:

$$\begin{aligned} G_{AB_3C_3}(t)Q_3G^T_{AB_3C_3} &= (e^{tA}-1)A^{-1}\left(A^T\right)^{-1}(e^{tA^T}-1) \\ &= (\gamma^2+\sigma^2)\left(e^{-2t\gamma}+1-2e^{-\gamma\tau}\cos\sigma t\right)I \end{aligned} \qquad (113)$$

which is identical to the previous term apart from a trivial scale factor. The second term in the sum, $G_{AB_2C_2}(t)Q_2G^T_{AB_2C_2}$, is rather complicated, and I have not been able to find a closed form for it.

Since $C_2=C_4=0$ the second and fourth block diagonal terms are proportional to q_μ and q_L respectively, and integrate to tq_μ and tq_L. The term $e^{tC_2}Q_2e^{tC_2^T}$ is identical to the "seasonal terms" found in UT1.

We now turn to the off diagonal terms. We start with the simplest terms, which are $G_{AB_1C_1}(t)Q_\mu e^{tC_1}$ and a similar term with subscript "3". The explicit form of the first of these is:

$$G_{AB_1C_1}(t)=e^{-\gamma t}\begin{pmatrix} -\cos\sigma t & \sin\sigma t \\ \sin\sigma t & \cos\sigma t \end{pmatrix}+\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (114)$$

with the following integrals

$$\begin{aligned} \int_0^t e^{-\gamma\tau}\sin(\sigma\tau)\,d\tau &= -\frac{e^{-\gamma t}\sigma\cos\sigma t+e^{-\gamma t}\gamma\sin\sigma t-\sigma}{\gamma^2+\sigma^2} \\ \int_0^t e^{-\gamma\tau}\cos(\sigma\tau)\,d\tau &= -\frac{e^{-\gamma t}\gamma\cos\sigma t-e^{-\gamma t}\sigma\sin\sigma t-\gamma}{\gamma^2+\sigma^2} \end{aligned} \qquad (115)$$

The contribution of this to the noise covariance can be evaluated completely.

The second term has the form:

$$G_{AB_3C_3}(t)=(\exp tA-1)A^{-1} \qquad (116)$$

which can be simplified.

7 Appendix: Exponential of 2x2 Matrix

Consider a 2x2 real matrix A:

$$A=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad (117)$$

I am interested in finding

$$\Phi(t)=\exp tA \qquad (118)$$

this can be decomposed as follows:

$$
\begin{aligned}
A &= \frac{a+d}{2}I + \frac{a-d}{2}T_1 + \frac{c+b}{2}T_2 + \frac{c-b}{2}T_3 &(119)\\
&\equiv A_0 I + \overrightarrow{A}\bullet\overrightarrow{T} &(120)
\end{aligned}
$$

where

$$
T_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \; T_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; T_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{121}
$$

are the Pauli spin matrices, apart from labeling and factors of i. Note that:

$$
T_1^2 = T_2^2 = I = -T_3^2 \tag{122}
$$

and that

$$
T_i T_j + T_j T_i = 0 \quad i \neq j. \tag{123}
$$

For any two matrices A and B that commute we have $\exp(A+B) = \exp A \exp B$. Since I commutes with everything, it follows that:

$$
\begin{aligned}
\exp tA &= \exp t(A_0 I + \overrightarrow{A}\bullet\overrightarrow{T}) &(124)\\
&= \exp tA_0 \exp t\overrightarrow{A}\bullet\overrightarrow{T} &(125)
\end{aligned}
$$

Hence we are left with evaluating:

$$
\exp t\overrightarrow{A}\bullet\overrightarrow{T} = \sum_{j=0} \frac{\left(t\overrightarrow{A}\bullet\overrightarrow{T}\right)^j}{j!} \tag{126}
$$

Note that by equation (122) we have the following simple identity:

$$
(\overrightarrow{A}\bullet\overrightarrow{T})^2 = A_1^2 + A_2^2 - A_3^2 \tag{127}
$$

There are two possibilities, depending on whether the term on the right hand side (RHS) of this equation is positive or negative. Suppose it is negative, so that we can write

$$
\omega = \sqrt{A_3^2 - A_1^2 - A_2^2} \tag{128}
$$

then

$$
\left(t\overrightarrow{A}\bullet\overrightarrow{T}\right)^{2j} = (-1)^j(\omega t)^{2j}. \tag{129}
$$

Using this, it is straightforward to evaluate the exponential. The sum naturally splits into a sum over the even and odd terms. For the even terms:

$$
\sum_{j=0} \frac{\left(t\overrightarrow{A}\bullet\overrightarrow{T}\right)^{2j}}{(2j)!} = \sum_{j=0} \frac{(-1)^j(\omega t)^{2j}}{(2j)!} = \cos\omega t \tag{130}
$$

While for the odd sum we have:

$$\sum_{j=0} \frac{\left(t\overrightarrow{A}\bullet\overrightarrow{T}\right)^{2j+1}}{(2j+1)!} = \sum_{j=0} \frac{(-1)^j (\omega t)^{2j} \left(t\overrightarrow{A}\bullet\overrightarrow{T}\right)}{(2j+1)!} = \frac{\sin \omega t}{\omega} \overrightarrow{A}\bullet\overrightarrow{T} \tag{131}$$

Hence, combining our results, we have:

$$\exp tA = \exp tA_0 \times \left(\cos \omega t \ + \frac{\sin \omega t}{\omega} \overrightarrow{A}\bullet\overrightarrow{T}\ \right). \tag{132}$$

which, in terms of the original elements of the matrix A is:

$$\exp tA = \exp t(\frac{a+d}{2}) \times \left(\cos \omega t + \frac{\sin \omega t}{\omega} \begin{pmatrix} \frac{a-d}{2} & b \\ c & \frac{d-a}{2} \end{pmatrix}\right) \tag{133}$$

If, on the other hand, we assume that

$$A_1^2 + A_2^2 - A_3^2 \geq 0 \tag{134}$$

then we can define

$$w = \sqrt{A_1^2 + A_2^2 - A_3^2} \tag{135}$$

and

$$\exp tA = \exp tA_0 \times \left(\cosh wt \ + \frac{\sinh wt}{w} \overrightarrow{A}\bullet\overrightarrow{T}\ \right). \tag{136}$$