Memo: Computation of a Doppler frequency shift for observations in the Solar system

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1 Coarse expression

It is deduced in the elementary coarse of physics that the ratio of the observed frequency f_o of the electromagnetic wave to the frequency f_e emitted by a body moving with the speed $\vec{v}_e$ with respect to the observer being at rest is

$$\frac{f_o}{f_e} = \frac{1 - \frac{1}{c}\,\vec{s}(t_e)\cdot\vec{v}_e(t_e)}{\sqrt{1 - \frac{v_e^2}{c^2}}} \tag{1}$$

Here $\vec{s}(t_e)$ is the unit vector of the direction to the emitter at the moment of coordinate time t_e when the light was emitted.

If the velocity of the emitter is given in the coordinate system at which the observer is not at rest, then the expression for the frequency ratio is changed to

$$\frac{f_o}{f_e} = \frac{1 - \frac{1}{c}\,\vec{s}(t_e)\cdot\vec{v}_e(t_e)}{1 - \frac{1}{c}\,\vec{s}(t_o)\cdot\vec{v}_o(t_o)}\,\frac{\sqrt{1 - \frac{v_o^2}{c^2}}}{\sqrt{1 - \frac{v_e^2}{c^2}}} \tag{2}$$

where t_o is the coordinate time of singal receving

Gravitating bodies also cause the frequency changes. Taking this effect into account and *assuming that the gravitating bodies are not moving*, modifies the previous equation this way:

$$\frac{f_o}{f_e} = \frac{1 - \frac{1}{c}\vec{s}(t_e)\cdot\vec{v}_e(t_e)}{1 - \frac{1}{c}\vec{s}(t_o)\cdot\vec{v}_o(t_o)}\,\frac{\sqrt{1 - \frac{v_o^2}{c^2}}}{\sqrt{1 - \frac{v_e^2}{c^2}}}\,\frac{1 - \sum\limits_i^n \frac{Gm_i}{c^2\,|\vec{r}_i(t_i) - \vec{r}_e(t_e)|}}{1 - \sum\limits_i^n \frac{Gm_i}{c^2\,|\vec{r}_i(t_i) - \vec{r}_o(t_o)|}} \tag{3}$$

where summation is done over all gravitating bodies. Assumption that the gravitating bodies are not moving results in $t_i = t_o$. At the first approximation taking into account motion of gravitating bodies can be done in the form of setting t_i to the retarded moment of time t_i', which is a solution of the gravitation null-cone equation:

$$t_i = t_o - \frac{1}{c}\Big|\vec{r}_i(t_i) - \vec{r}_o(r_o)\Big| \tag{4}$$

I do not have by hand estimates of errors of such an approximation.

2 Refined expression

[Kopeikin and Shaeffer, 1999] derived a rigorous expression for the case when gravitating bodies have an arbitrary motion.

[to be added in the future.]

3 Computation of the Doppler frequency shift using the coarse expression

We can re-write expression 3 in this way:

$$\frac{f_o}{f_e} = \frac{1-a_e}{1-a_o}\,\frac{\sqrt{1-b_o^2}}{\sqrt{1-b_e^2}}\,\frac{1-c_e^2}{1-c_o^2} \tag{5}$$

where

$$\begin{aligned}
a_k &= \tfrac{1}{c}\vec{s}(t_k)\cdot\vec{v}_k(t_k) \\
b_k &= \tfrac{|v_k|}{c} \\
c_k^2 &= \sum_i^n \frac{Gm_i}{c^2\,|\vec{r}_i(t_i) - \vec{r}_k(t_k)|}
\end{aligned}$$

Expand expression 5 discarding terms of $O(c^{-4})$:

$$\begin{aligned}
D = \frac{f_o}{f_e} - 1 \;=\;& \quad a_o - a_e \\
& + a_o^2 - a_e a_o - \tfrac{1}{2}b_o^2 + \tfrac{1}{2}b_e^2 - c_e^2 + c_o^2 \\
& + a_o^3 - a_e a_o^2 - \tfrac{1}{2}a_o b_o^2 + \tfrac{1}{2}a_o b_e^2 + \tfrac{1}{2}a_e b_o^2 - \tfrac{1}{2}a_e b_e^2 - a_o c_e^2 + a_e c_e^2 + a_o c_o^2 - a_e c_o^2
\end{aligned} \tag{6}$$

According to [Brunmberg, 1999], the velocity of the observer with repsect to the geoctenter, $\vec{v}_g$, is related to its velocity with respect to the barycenter, $\vec{v}_b$, as

$$\vec{v}_b = \frac{\sqrt{1-\frac{V_\oplus^2}{c^2}}}{1+\frac{\vec{v}_g\cdot\vec{V}_\oplus}{c^2}}\left\{\vec{v}_g \;+\; \left[\left(\frac{1}{\sqrt{1-\frac{V_\oplus^2}{c^2}}} - 1\right)\frac{\vec{v}_g\cdot\vec{V}_\oplus}{V_\oplus^2} \;+\; \frac{1}{\sqrt{1-\frac{V_\oplus^2}{c^2}}}\right]\vec{V}_\oplus\right\} \tag{7}$$

Expanding the expression 7 in series over 1/c retaining terms of $1/c^2$ and adding contribution to the metric due to gravitational potential, we get the following transformation:

$$\vec{v}_o = \left(1 - \left(\frac{U_\odot}{c^2} - L_b\right) - \frac{1}{2}\frac{V_\oplus^2}{c^2} - \frac{\vec{v}_g\cdot\vec{V}_\oplus}{c^2}\right)\vec{v}_g \;+\; \left(1 - \frac{1}{2c^2}\vec{v}_g\cdot\vec{V}_\oplus\right)\vec{V}_\oplus \;+\; O(c^{-4}) \tag{8}$$

where L_b is the metric parameter, $L_b = 1.55051976772\cdot 10^{-8}$, if the TDB metric is used.

References

[Brunmberg, 1999] Brumberg, V.A., Essential relativistic mechanics, 274p, 1991.

[Kopeikin and Shaeffer, 1999] Kopeikin, S.M. and Shaeffer G., Physical Review, D, vol. 50, 124002, 1999.