Memo: The number of independent equations of conditions in VLBI observations

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2005.10.18

If N stations observed a source during a scan, the number of group delay determined by a correlator and post-correlator software is $N(N-1)/2$. We can write $N(N-1)/2$ equations of conditions. However, only $N-1$ of these equations are linearly independent. Other equations do not contribute to parameter estimation. Currently, the Calc/Solve and, presumably, other software packages, when computes statistics counts all equations as independent. However, the number of linearly independent equations in k scans which contribute to parameter estimation is only $N_{li} = \sum\limits_k (n_k - 1)$.

Let's consider a three station network: station 1, station 2, station 3. Three delays τ_{ij} and its time derivatives are determined: $\tau_{12}(t_1), \tau_{13}(t_1), \tau_{23}(t_2), \dot{\tau_{12}}, \dot{\tau_{13}}, \dot{\tau_{23}}$. Notation $\tau_{ij}(t_i)$ means delay defined as the difference between two quantities: interval of proper time measured by clock of the i station between events coming the wavefront to the i station and clock synchronization and the interval of proper time measured by clock of the j station between events coming the wavefront to the j station and clock synchronization. t_i is a time tag of event of coming the wavefront to the i station at TAI or TDT.

The fundamental equation of VLBI time delay reads in this form:

$$\tau_{ij}(t_i) = \frac{1}{c} \frac{\hat{\mathcal{E}}(e(t_i))\,(\vec{r}_j - \vec{r}_i)\,\vec{s}}{1 + \frac{1}{c}\left(\vec{V_\oplus}(t_i) + \frac{\partial}{\partial t}\hat{\mathcal{E}}(e(t_i))\,\vec{r}_i\right)} + \frac{1}{c^2}\vec{V_\oplus}(t_i)\,(\vec{r}_j - \vec{r}_i) + O(GM/c^3, 1/c^3) \tag{1}$$

where $\hat{\mathcal{E}}(e(t))$ — the matrix of the Earth rotation which depends on a three-dimensional function $e(t)$, $\vec{r}_k$ — vector of position of the kth station in a crust fixed coordinate system, $\vec{s}$ is a vector of source position in a barycentric coordinate system, $\vec{V_\oplus}$ is a vector of the Earth velocity in the barycentric coordinate system.

Direct check shows validity of the delay closure equation:

$$\tau_{12}(t_1) - \tau_{13}(t_1) + \tau_{23}(t_2) + \dot{\tau}_{23}(t_2)\,\tau_{12}(t_1) = 0 \tag{2}$$

In equation (1) time delay is a function of four multi-dimensional variables: $\tau_{ij}(t_i) = f(e(t_i), \vec{r}_k(t_i), \vec{s}(t_i), \vec{V_\oplus}(t_i))$. Let us consider an extended model

$$\tau^{obs}_{ij}(t_i) = \tau_{ij}\left(e(t_i), \vec{r}_k(t_i), \vec{s}(t_i), \vec{V_\oplus}(t_i)\right) + F_i(\vec{r}_i, t) - F_j(\vec{r}_j, t) \tag{3}$$

where τ_{ij} satisfies equation (1), and $F_k(\vec{r}_k, t)$ is a function which depends on position of the k-th and on time and takes into account a contribution of station-dependent delays. Let us consider the Earth orientation parameters, vector of station coordinates and source positions, $e, \vec{r}_k, \vec{s}$, as functions which depend on vectors of parameters $\vec{p}_a, \vec{p}_b, \vec{p}_c$:

$$\begin{array}{lcl} e & = & f_a(\vec{p}_a, t) \\ \vec{r}_k & = & f_b(\vec{p}_b, t) \\ \vec{s} & = & f_c(\vec{p}_c, t) \end{array} \tag{4}$$

In a similar way we can consider $F_k(\vec{r}_k, t) = f_d(\vec{p}_d, t)$ as depending on a vector of parameters $\vec{p}_d$.

Lemma 1 *Let us have three measurement of group delay and its time derivatives at stations 1, 2 and 3: $\tau_{12}(t_1), \tau_{13}(t_1), \tau_{23}(t_2), \dot{\tau_{12}}, \dot{\tau_{13}}, \dot{\tau_{23}}$ which are related to the same scan. Let us determine adjustments to parameters $\vec{p}_a, \vec{p}_b, \vec{p}_c, \vec{p}_d$ in the vicinity of their apriori values using the linearized least squares (LSQ) method. Then the number of linearly independent equations of conditions is 2.*

Proof

First of all notice that if to make a substitution $\tau_{23}(t_1) = \tau_{23}(t_2) + \dot{\tau_{23}}(t_2)\tau_{12}$. in our equation $\tau_{23}(t_2)$, then the delay closure equation becomes more simple:

$$\tau_{12}(t_1) - \tau_{13}(t_1) + \tau_{23}(t_1) = 0 \tag{5}$$

Second, notice that direct substitution of the relationship $\tau_{23}(t_1) = \tau_{23}(t_2) + \dot{\tau_{23}}(t_2)\tau_{12}$ into the fundamental equation for VLBI delay (1) gives us

$$\tau_{23}(t_1) = \frac{1}{c} \frac{\hat{\mathcal{E}}(e(t_1))\,(\vec{r}_3 - \vec{r}_2)\,\vec{s}}{1 + \dfrac{1}{c}\left(\vec{V}_\oplus(t_1) \,+ \dfrac{\partial}{\partial t}\hat{\mathcal{E}}(e(t_1))\,\vec{r}_1\right)} + \frac{1}{c^2}\vec{V}_\oplus(t_1)\,(\vec{r}_3 - \vec{r}_2) + O(GM/c^3, 1/c^3) \tag{6}$$

Then let us show that the equations of conditions E_{12}, E_{13}, E_{23} satisfy relationship $E_{12} - E_{13} + E_{23} = 0$ for all groups of parameters, i.e. are linearly dependent.

1. Case of $\vec{p}_d$. Three equations of conditions are as follows:

$$\begin{array}{lclll} E_{12} & = & \dfrac{\partial F_1}{\partial \vec{p}_{d1}} & -\dfrac{\partial F_2}{\partial \vec{p}_{d2}} & \\ E_{13} & = & \dfrac{\partial F_1}{\partial \vec{p}_{d1}} & & -\dfrac{\partial F_3}{\partial \vec{p}_{d3}} \\ E_{23} & = & & \dfrac{\partial F_2}{\partial \vec{p}_{d2}} & -\dfrac{\partial F_3}{\partial \vec{p}_{d3}} \end{array}$$

   where $\vec{p_d k}$ is the group of parameters which is related to the kth station. Adding the first equation to the third and subtracting the second one, we check that $E_{12} - E_{13} + E_{23} = 0$.

2. Case of $\vec{p}_a$. Three equations of conditions are as follows:

$$E_{12} = \frac{1}{c}\frac{\dfrac{\partial\hat{\mathcal{E}}}{\partial e}\dfrac{\partial e}{\partial \vec{p}_a}(\vec{r}_1-\vec{r}_2)\vec{s}}{1+\dfrac{1}{c}\left(\vec{V}_\oplus(t_1)\ +\dfrac{\partial}{\partial t}\hat{\mathcal{E}}(e(t_1))\,\vec{r}_1\right)}$$

$$E_{13} = \frac{1}{c}\frac{\dfrac{\partial\hat{\mathcal{E}}}{\partial e}\dfrac{\partial e}{\partial \vec{p}_a}(\vec{r}_1-\vec{r}_3)\vec{s}}{1+\dfrac{1}{c}\left(\vec{V}_\oplus(t_1)\ +\dfrac{\partial}{\partial t}\hat{\mathcal{E}}(e(t_1))\,\vec{r}_1\right)}$$

$$E_{23} = \frac{1}{c}\frac{\dfrac{\partial\hat{\mathcal{E}}}{\partial e}\dfrac{\partial e}{\partial \vec{p}_a}(\vec{r}_2-\vec{r}_3)\vec{s}}{1+\dfrac{1}{c}\left(\vec{V}_\oplus(t_1)\ +\dfrac{\partial}{\partial t}\hat{\mathcal{E}}(e(t_1))\,\vec{r}_1\right)}$$

Adding the first equation to the third and subtracting the second one, we check that $E_{12} - E_{13} + E_{23} = 0$.

3. Case of $\vec{p}_c$. Three equations of conditions are as follows:

$$E_{12} = \frac{1}{c}\frac{\hat{\mathcal{E}}(e(t_1))\,(\vec{r}_1-\vec{r}_2)\,\dfrac{\partial\vec{s}}{\partial\vec{p}_c}}{1+\dfrac{1}{c}\left(\vec{V}_\oplus(t_1)\ +\dfrac{\partial}{\partial t}\hat{\mathcal{E}}(e(t_1))\,\vec{r}_1\right)}$$

$$E_{13} = \frac{1}{c}\frac{\hat{\mathcal{E}}(e(t_1))\,(\vec{r}_1-\vec{r}_3)\,\dfrac{\partial\vec{s}}{\partial\vec{p}_c}}{1+\dfrac{1}{c}\left(\vec{V}_\oplus(t_1)\ +\dfrac{\partial}{\partial t}\hat{\mathcal{E}}(e(t_1))\,\vec{r}_1\right)}$$

$$E_{23} = \frac{1}{c}\frac{\hat{\mathcal{E}}(e(t_1))\,(\vec{r}_2-\vec{r}_3)\,\dfrac{\partial\vec{s}}{\partial\vec{p}_c}}{1+\dfrac{1}{c}\left(\vec{V}_\oplus(t_1)\ +\dfrac{\partial}{\partial t}\hat{\mathcal{E}}(e(t_1))\,\vec{r}_1\right)}$$

Again, $E_{12} - E_{13} + E_{23} = 0$.

4. Case of $\vec{p}_b$. Three equations of conditions are as follows:

$$\begin{aligned}
E_{12} &= \frac{\partial\tau_{12}}{\partial\vec{r}_1}\frac{\partial\vec{r}_1}{\partial\vec{p}_b} + \frac{\partial\tau_{12}}{\partial\vec{r}_2}\frac{\partial\vec{r}_2}{\partial\vec{p}_b} \\
E_{13} &= \frac{\partial\tau_{13}}{\partial\vec{r}_1}\frac{\partial\vec{r}_1}{\partial\vec{p}_b} + \frac{\partial\tau_{13}}{\partial\vec{r}_3}\frac{\partial\vec{r}_3}{\partial\vec{p}_b} \\
E_{23} &= \frac{\partial\tau_{23}}{\partial\vec{r}_1}\frac{\partial\vec{r}_1}{\partial\vec{p}_b} + \frac{\partial\tau_{23}}{\partial\vec{r}_2}\frac{\partial\vec{r}_2}{\partial\vec{p}_b} + \frac{\partial\tau_{13}}{\partial\vec{r}_3}\frac{\partial\vec{r}_3}{\partial\vec{p}_b}
\end{aligned}$$

It is easy to check that

$$\frac{\partial \tau_{12}}{\partial \vec{r}_2} = \frac{1}{c}\frac{(\hat{\mathcal{E}}(e)\,\vec{i}\,)\,\vec{s}}{1+\frac{1}{c}\left(\vec{V}_{\oplus}(t_1)\,+\frac{\partial}{\partial t}\hat{\mathcal{E}}(e(t_1))\,\vec{r}_1\right)} + \frac{1}{c^2}(\vec{V}_{\oplus}\,\vec{i}) \tag{7}$$

$$\frac{\partial \tau_{23}}{\partial \vec{r}_2} = -\frac{1}{c}\frac{(\hat{\mathcal{E}}(e)\,\vec{i}\,)\,\vec{s}}{1+\frac{1}{c}\left(\vec{V}_{\oplus}(t_1)\,+\frac{\partial}{\partial t}\hat{\mathcal{E}}(e(t_1))\,\vec{r}_1\right)} - \frac{1}{c^2}(\vec{V}_{\oplus}\,\vec{i}) \tag{8}$$

where $\vec{i}$ is the identity vector:

$$\vec{i} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Therefore, $\frac{\partial \tau_{12}}{\partial \vec{r}_2} = -\frac{\partial \tau_{23}}{\partial \vec{r}_2}$

$$\begin{aligned}
\frac{\partial \tau_{12}}{\partial \vec{r}_1} &= -\frac{1}{c}\frac{(\hat{\mathcal{E}}(e)\,\vec{i}\,)\,\vec{s}}{1+\frac{1}{c}\left(\vec{V}_{\oplus}(t_1)\,+\frac{\partial}{\partial t}\hat{\mathcal{E}}(e(t_1))\,\vec{r}_1\right)} - \frac{1}{c^2}(\vec{V}_{\oplus}\,\vec{i}) - \frac{1}{c^2}\frac{\left(\frac{\partial}{\partial t}\hat{\mathcal{E}}(e)\;\vec{i}\right)\,\hat{\mathcal{E}}(e)(\vec{r}_2-\vec{r}_1)\,\vec{s}}{1+\frac{1}{c}\left(\vec{V}_{\oplus}(t_1)\,+\frac{\partial}{\partial t}\hat{\mathcal{E}}(e(t_1))\,\vec{r}_1\right)} \\
\frac{\partial \tau_{13}}{\partial \vec{r}_1} &= \frac{1}{c}\frac{(\hat{\mathcal{E}}(e)\,\vec{i}\,)\,\vec{s}}{1+\frac{1}{c}\left(\vec{V}_{\oplus}(t_1)\,+\frac{\partial}{\partial t}\hat{\mathcal{E}}(e(t_1))\,\vec{r}_1\right)} + \frac{1}{c^2}(\vec{V}_{\oplus}\,\vec{i}) + \frac{1}{c^2}\frac{\left(\frac{\partial}{\partial t}\hat{\mathcal{E}}(e)\;\vec{i}\right)\,\hat{\mathcal{E}}(e)(\vec{r}_3-\vec{r}_1)\,\vec{s}}{1+\frac{1}{c}\left(\vec{V}_{\oplus}(t_1)\,+\frac{\partial}{\partial t}\hat{\mathcal{E}}(e(t_1))\,\vec{r}_1\right)} \\
\frac{\partial \tau_{23}}{\partial \vec{r}_1} &= \frac{1}{c^2}\frac{\left(\frac{\partial}{\partial t}\hat{\mathcal{E}}(e)\;\vec{i}\right)\,\hat{\mathcal{E}}(e)(\vec{r}_3-\vec{r}_2)\,\vec{s})}{1+\frac{1}{c}\left(\vec{V}_{\oplus}(t_1)\,+\frac{\partial}{\partial t}\hat{\mathcal{E}}(e(t_1))\,\vec{r}_1\right)}
\end{aligned}$$

Again, $E_{12} - E_{13} + E_{23} = 0$.

The lemma is proven.

Lemma 2

The number of independent closure equations $e_{ij} - e_{ik} + e_{jk}$ which can be built at baselines from N different points is $(N-1)(N-2)/2$

Proof

The formula is true if $N = 3$. Let us have k points. Adding the $k+1$th point adds k baselines and $k(k-1)/2$ new triangles. Let us count vortices 0, the new, $k+1$th station, 1, 2, ..., k. Let us make a sequence of $k-1$ equations $e_{01} - e_{0i} + e_{1i} = 0$. These equations are linearly independent, since a unique element e_{0i} enters each equation. Any other equations between the vortices 0, a, b, where $a > 1, b \neq a$ are not independent:

$$\begin{array}{ccccccc}
e_{01} & - & e_{0a} & + & e_{1a} & = & 0 \\
e_{01} & - & e_{0b} & + & e_{1b} & = & 0 \\
e_{0a} & - & e_{0b} & + & e_{ab} & = & 0
\end{array} \tag{9}$$

The last equation follows from $e_{1a} - e_{1b} + e_{ab} = 0$ and is not independent. Thus, adding the $k+1$th station adds $k-1$ independent closure equations. Invoking the principle of mathematical induction, we *prove the lemma.*

Theorem *Let us have measurement of group delays and their time derivatives during the same scan at all baselines between N different stations. Let us determine adjustments to parameters $\vec{p}_a, \vec{p}_b, \vec{p}_c, \vec{p}_d$ in the vicinity of their apriori values using the linearized least squares (LSQ) method. Then the number of linearly independent equations of conditions is $N-1$.*

Proof

According to the lemma 1, the closure conditions for equations of conditions are satisfied for any triplet of station. Lemma 2 states that the number of independent closure conditions is $(N-1)(N-2)/2$. The total number of equations of conditions equals to the number of baselines, $N(N-1)/2$. Then the number of linearly independent equation of conditions is $N(N-1)/2 - (N-1)(N-2)/2 = N-1$.
The theorem is proved.

Corollary

The portion of the system of equations of conditions which corresponds to observations of the same scan can be written this way using equivalent transformations:

$$
\begin{array}{lllllllll}
a_{p1}x_1 + a_{p2}x_2 + a_{p3}x_3 + \ldots & = & \tau_p^{obs} - \tau_p^{thr} & + n_i & - n_j & + n_{ij} & & \\
a_{q1}x_1 + a_{q2}x_2 + a_{q3}x_3 + \ldots & = & \tau_q^{obs} - \tau_q^{thr} & + n_f & - n_g & + n_{fg} & & \\
\ldots & & & & & & & \\
0 & = & \tau_{ij}^{obs} & - \tau_{ik}^{obs} & + \tau_{jk}^{obs} & + n_{ij} & - n_{ik} & + n_{jk} \\
0 & = & \tau_{fg}^{obs} & - \tau_{fh}^{obs} & + \tau_{gh}^{obs} & + n_{fg} & - n_{fh} & + n_{gh} \\
\ldots & & & & & & &
\end{array}
\tag{10}
$$

The first part consists of $N-1$ equations with linearly independent left hand side parts, the second part consists of $(N-1)(N-2)/2$ redundant observations which do not depend on parameters. n_i — is a station-dependent constituent of the noise process, n_{ij} — is a baseline-dependent constituent of the noise process.

The linearly dependent part of the system of equations of conditions (10) states that the misclosure of VLBI delays is determined entirely by baseline dependent noise. No parameters in the groups $\vec{p}_a, \vec{p}_b, \vec{p}_c, \vec{p}_d$ affect it. So, these equations should be ignored in a LSQ solution and in computation of solution statistics.

It can be easily shown that all parameters which are currently estimated with using VLBI observations, pole coordinates, their rate, nutation daily offset, amplitudes of harmonic EOP variations, site position and velocities, their harmonic and aharmonic variations, source coordinates, proper motions, parallaxes, station-dependent clock function, troposphere zenith path delay, tilt angles of the atmosphere symmetry axis, fall in the category of $\vec{p}_a, \vec{p}_b, \vec{p}_c$, or $\vec{p}_d$. The only exception is estimation of so-called baseline-dependent clock offsets. Baseline-dependent clock do not have physical meaning, and with rare exceptions related to failure of fringe searching process have zero estimates.